TANNAKA-KREIN DUALITY FOR HOPF ALGEBROIDS

BY

PHÙNG HÔ HAI

Institute of Mathematics
18 Hoang Quoc Viet, 10307, Hanoi, Vietnam
e-mail: phung@math.ac.vn
Herrn Prof. B.Pareigis zum 65 Geburtstag gewidmet

ABSTRACT

We show that a Hopf algebroid can be reconstructed from a monoidal functor from a monoidal category into the category of rigid bimodules over a ring. We study the equivalence between the original category and the category of comodules over the reconstructed Hopf algebroid.

Introduction

The Tannaka-Krein duality asserts that a compact group can be uniquely determined by the category of its finite dimensional unitary representations. Many efforts have been made in generalizing this result, which also advance the development of many branches of mathematics, such as C^*-algebras, harmonic analysis, algebraic geometry. Especially, Tannaka-Krein duality was also one of the sources of quantum groups.

The algebraic version of this theory was suggested by A. Grothendieck and developed by Saavedra [16] and Deligne [2]. An important result of the algebraic Tannaka-Krein theory is a theorem of Deligne, developing Saavedra's ideas. It states that there is a dictionary between tensor categories over a field k together with an exact tensor functor (fiber functor) to the category of quasi-coherent sheaves over a k-scheme S and transitive groupoids over S.

The proof of Tannaka-Krein duality is divided into two parts, the (re)construction theorem which aims to reconstruct the group from the category of its representations and the representation theorem which aims to prove the equivalence between the original category and the category of representations of the reconstructed group.

The idea of the reconstruction theorem was one of the motivations for Quantum Groups. From this point of view a rigid monoidal category (without a symmetry) corresponds to a quantum group. In fact, one can construct from a rigid monoidal category together with a monoidal functor into the category of finite-dimensional vector spaces (over k) a k-Hopf algebra, which is understood as the "function algebra" over a quantum group. This idea was first proposed in the work of Lyubashenko [8]. The Tannaka-Krein duality for compact quantum groups was proved by Woronowicz [21]. Majid obtained the reconstruction theorem in a more general setting of a monoidal category and a monoidal functor in to another braided monoidal category, see [12] and references therein.

In this more general setting, a Hopf algebra (in a braided monoidal category) cannot be reconstructed from its representation (comodule) category. Usually, the reconstructed Hopf algebra is bigger, see, for instance, [15]. Lyubashenko [9] suggests reconstructing the Hopf algebra not lying in the target category but rather in its tensor square. McCrudden [14] generalizes the duality to the setting of higher categories.

An important ingredient of the Tannaka-Krein duality is the fiber functor. One might ask, for which kind of monoidal categories do such functors exist. An answer to this question can be called an embedding theorem. For a tensor category over a field k of characteristic 0, P. Deligne [2] gave an interesting criterion in terms of the categorical dimension. A parallel result for C^* tensor categories was given by Doplicher and Roberts [3].

In our previous work [4] an embedding theorem for arbitrary (rigid) monoidal categories was given, but the embedding goes into the category of bimodules over a ring. This result raises the problem of Tannaka-Krein duality for functors with the category of bimodules over a ring as the target. However, according to Schauenburg [18], it is generally impossible to construct a braiding in a bimodule category, so one cannot apply Majid results to reconstruct a Hopf algebra in this category.

It turns out that one can reconstruct from the above data a Hopf algebroid in the sense of Takeuchi, Lu and Schauenburg [19, 20, 7, 17]. This construction is in a sense analogous to those of Lyubashenko [9] and Mccrudden [14].

Combining the Tannaka-Krein duality done here and the embedding theorem of [4], we can realize a rigid category as the comodule category over a Hopf algebroid defined over a certain ring (Corollary 2.2.7).

The paper is constructed as follows. In Section 1, we recall the notion of Hopf algebroids defined over a ring. We define bialgebroid as a monoidal object in the monoidal category of coalgebroids. Next, we recall a notion of antipode and Hopf algebroids and prove some basic facts on dual comodules over Hopf algebroids. This is the more technically difficult part of the work. In fact, there are at least two definitions of antipode on a bialgebroid [7, 17]. In [7] the definition of the antipode more or less imitates the usual antipode, in [17] the antipode is defined as a condition for the coincidence of internal hom-functors in the category of modules and in the underlying category of R-bimodules. Our motivation for the antipode is the condition for the existence of dual comodules over a bialgebroid. It is somewhat unexpected that the antipode introduced by Schauenburg in [17] while studying duals of modules over a bialgebroid fits well into our frame-work.

In Section 2, we prove the Tannaka-Krein duality for Hopf algebroids. Some embedding and reconstruction results were also obtained by Hayashi [5, 6] for face algebras, which were shown by Schauenburg to be a special case of Hopf algebroids. Our result here is a generalization of Hayashi's result.

1. Hopf algebroids and its comodules

Except for some results in Subsections 1.4 and 1.8, the materials of this section are known, they can be found in [20, 7, 23, 17].

First we review some basic notions of rings and corings over an associative ring. In Section 1.2 we recall the notion of coalgebroids which was first studied by Takeuchi [20]. In Subsections 1.4 and 1.5 we study comodules over a coalgebroid and prove some lemmas which will be needed in the sequel. In Subsection 1.6 we recall the notion of bialgebroids in the sense of [20, 7]. In 1.7 we define the tensor product of two comodules over a bialgebroid and in 1.8 the dual to a comodule.

1.1. R-RINGS AND R-CORINGS. Let us fix a commutative ring k. Throughout this paper, we will be working in the category of k-modules, in other words, we shall assume that everything is k-linear.

Let R be an algebra over k, which will usually be fixed. The object of our study is the category R-Bimod of R-bimodules. In this category, there is a monoidal structure with the tensor product being the usual tensor product over R. This tensor product is closed in the sense that there exist the right adjoint functors to the functors $M \otimes_R -$ and $- \otimes_R M$ for all R-bimodules M, given by $\mathrm{Hom}_R(M, -)$ and $_R\mathrm{Hom}(M, -)$, where $\mathrm{Hom}_R(-, -)$ (resp., $_R\mathrm{Hom}(-, -)$) denotes the set of R-linear maps with respect to the right (resp., left) actions of R.

Having the monoidal structure on R-Bimod we define R-rings and R-corings as monoids and comonoids in this category. The data for an R-ring consist of an $R - R$-linear map $m : A \otimes_R A \longrightarrow A$, called product, and an $R - R$-linear map $u : R \longrightarrow A$ called unit, satisfying the usual associativity and unity properties. Set $1_A := u(1_R)$ and denote $m(a \otimes b)$ by $a \cdot b$, then A is a k-algebra in the usual sense. We notice that if R is commutative and A is an algebra over R in the usual sense then it is an R-ring in our sense but the converse is not true since the image of R under u is generally not in the center of A. In fact, any (associative) k-algebra homomorphism $R \longrightarrow A$ induces a structure of R-ring over A.

R-corings are defined in the dual way. A structure of R-coring over an R-bimodule C consists of an $R - R$-linear map $\Delta : C \longrightarrow C \otimes_R C$ called **coproduct** and an $R - R$-linear map $\varepsilon : C \longrightarrow R$, called **counit**, satisfying the usual coassociativity and counity axioms. We shall use Sweedler's notation for denoting the coproduct:

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}.$$

A right C-comodule is a right R-module M equipped with an R-linear coaction $\delta : M \longrightarrow M \otimes_R C, \delta(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$, satisfying

$$\sum_{(m)} \delta(m_{(0)}) \otimes m_{(1)} = \sum_{(m)} m_{(0)} \otimes \Delta(m_{(1)}) \quad \text{and} \quad \sum_{(m)} m_{(0)} \otimes \varepsilon(m_{(1)}) = m$$

(having in mind the identification $M \otimes_R R \cong M$). Notice that the R-linearity of δ means $\delta(mr) = \sum_{(m)} m_{(0)} \otimes m_{(1)}r$.

1.2. R-COALGEBROIDS. We consider in this subsection the category R–R-Bimod of double R-bimodules. That is, we will have two R-bimodule structures on a k-module, which commute with each other. To distinguish the two structures we will denote the first one by σ and the second one by τ. Thus, we have four actions:

$$R \otimes_k M \longrightarrow M; \quad r \otimes M \longmapsto \sigma(r)m, \quad M \otimes_k R \longrightarrow M; \quad m \otimes r \longmapsto m\sigma(r),$$
$$R \otimes_k M \longrightarrow M; \quad r \otimes M \longmapsto \tau(r)m, \quad M \otimes_k R \longrightarrow M; \quad m \otimes r \longmapsto m\tau(r),$$

There are several possibilities to take tensor products over R of $R - R$-bimodules. We use the notation $M\ _\tau\otimes^\sigma N$ for the tensor product with respect to the right action of M by τ and the left action on N by σ, i.e.,

$$M\ _\tau\otimes^\sigma N := M \otimes_k N \Big/ \Big(m\tau(r) \otimes n = m \otimes \sigma(r)n \Big).$$

Here, the letters σ and τ on the two sides of the tensor sign denote correspondingly the actions taken in the definition of the tensor product. Other tensor products will be denoted in a similar way. The rule for notation is that the left action will be placed in the upper place and the right action will be placed in the lower place on the two sides of the tensor sign.

For the tensor product $M\ _\tau\otimes^\sigma N$ we specify the following actions to make it an object in $R - R$-Bimod:

$$\tau(a)(h \otimes k) = h \otimes \tau(a)k, \quad (h \otimes k)\tau(a) = h \otimes k\tau(a),$$
$$\sigma(a)(h \otimes k) = \sigma(a)h \otimes k, \quad (h \otimes k)\sigma(a) = h\sigma(a) \otimes k.$$

Here we adopt the convention that the action of R has preference over the tensor product.

Definition ([20]): An R-**coalgebroid** is an $R - R$-bimodule L equipped with k-linear maps $\Delta : L \longrightarrow L\ _\tau\otimes^\sigma L$, called coproduct, and $\varepsilon : L \longrightarrow R$, called counit, satisfying the following conditions:

(i) Δ is a morphism in $R - R$-bimod and (the coassociativity):

$$(\mathrm{id}_L\ _\tau\otimes^\sigma \Delta)\Delta = (\Delta\ _\tau\otimes^\sigma \mathrm{id}_L)\Delta;$$

(ii) ε satisfies (the linearity with respect to the actions of R)

$$\varepsilon(\sigma(a)h\tau(b)) = a\varepsilon(h)b,$$

and (the counity)

$$(\varepsilon\ _\tau\otimes^\sigma \mathrm{id}_L)\Delta = (\mathrm{id}_L\ _\tau\otimes^\sigma \varepsilon)\Delta = \mathrm{id}_L;$$

(iii) moreover, ε satisfies the following condition

$$\varepsilon(\tau(a)h) = \varepsilon(h\sigma(a)).$$

Note that, by definition, ε is not necessarily a morphism of $R - R$-bimodules.

We shall use Sweedler's notation for the coproduct: $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$. The linearity of Δ now reads:

$$(1.1) \qquad \Delta(\tau(a)\sigma(b)h\tau(c)\sigma(d)) = \sum_{(h)} \sigma(b)h_{(1)}\sigma(d) \otimes \tau(a)h_{(2)}\tau(c).$$

Analogously, the counity condition has the following form

$$(1.2) \qquad \sum_{(h)} \sigma\varepsilon(h_{(1)})h_{(2)} = \sum_{(h)} h_{(1)}\tau\varepsilon(h_{(2)}) = h.$$

Combining these equations, we have the following identities

$$(1.3) \qquad h\sigma(a) = \sum_{(h)} \sigma\varepsilon\left(h_{(1)}\sigma(a)\right) h_{(2)}; \quad \tau(a)h = \sum_{(h)} h_{(1)}\tau\varepsilon\left(\tau(a)h_{(2)}\right),$$

whence

$$(1.4) \qquad \varepsilon\left(\tau(a)h\sigma(b)\right) = \sum_{(h)} \varepsilon(h_{(1)}\sigma(b))\varepsilon\left(\tau(a)h_{(2)}\right).$$

Remark: The condition in (iii) can be replaced by the following (cf. [20, §3])

$$(1.5) \qquad \sum_{(h)} \tau(a)h_{(1)} \, {}_\tau\otimes^\sigma h_{(2)} = \sum_{(h)} h_{(1)} \, {}_\tau\otimes^\sigma h_{(2)}\sigma(a).$$

This condition is equivalent to the existence of an anchor as in [23, 7]. In fact, the algebra $\mathrm{End}_k(R)$ has a structure of an $R - R$-bimodule, we specify it as follows:

$$\left(\sigma(a)f\sigma(b)\right)(c) := af(bc), \quad \left(\tau(a)f\tau(b)\right)(c) = f(ca)b.$$

Then, the counity induces a morphism of $R - R$-bimodules $\eta: L \longrightarrow \mathrm{End}_k(R)$, given by $\eta(h)(a) := \varepsilon(\tau(a)h) = \varepsilon(h\sigma(a))$. The map η is called an anchor [23, 7] (this map is generally different from a map, also denoted by η, introduced in [20, §3]).

1.3. AN EXAMPLE. Let M be a right R-module. Then M^* is a left R-module with the action given by

$$(r\varphi)(m) := r(\varphi(m)); \quad r \in R, m \in M, \varphi \in M^*.$$

If M is finitely generated (f.g.) projective, M is a direct summand of $R^{\oplus d}$ considered as right module over R, then M^* is a direct summand of $R^{\oplus d}$ considered as left module over R. Further, if we fix a generating set $m_1, m_2, \ldots, m_d$ given by the projection from $R^{\oplus d}$ then we can find a generating set $\varphi^1, \varphi^2, \ldots, \varphi^d$ for M^* such that for any $m \in M$, the following equation holds

$$(1.6) \qquad\qquad m = \sum_{i=1}^{d} m_i \varphi^i(m).$$

We define the following map

$$(1.7) \qquad\qquad \mathrm{ev}_{k,M} : M^* \otimes_k M \to R; \quad \varphi \otimes m \mapsto \varphi(m),$$

$$(1.8) \qquad\qquad \mathrm{db}_{k,M} : k \to M \otimes_R M^*; \quad 1 \mapsto \sum_i m_i \otimes \varphi^i.$$

Notice that $\mathrm{ev}_{k,M}$ is a morphism of R-bimodules: $\mathrm{ev}_{k,M}(r\varphi \otimes ms) = (r\varphi)(ms) = r\varphi(m)s$. The equation in (1.6) implies the following relations for $\mathrm{ev} = \mathrm{ev}_{k,M}$ and $\mathrm{db} = \mathrm{db}_{k,M}$

$$(1.9) \quad (\mathrm{ev} \otimes_R \mathrm{id}_{M^*})(\mathrm{id}_{M^*} \otimes_k \mathrm{db}) = \mathrm{id}_{M^*}; \quad (\mathrm{id}_M \otimes_R \mathrm{ev})(\mathrm{db} \otimes_k \mathrm{id}_M) = \mathrm{id}_M.$$

Conversely, if there exists to a right R-module M a left R-module $M^\vee$ and morphisms $\mathrm{ev} : M^\vee \otimes_k M \to R$ and $\mathrm{db} : k \to M \otimes_R M^\vee$, satisfying the identities in (1.9) then R is f.g. projective. Indeed, we have by means of (1.9) the following natural isomorphism

$$\mathrm{Hom}_R(P \otimes_k M, N) \cong \mathrm{Hom}_k(P, N \otimes_R M^\vee); \quad f \mapsto (f \otimes_R \mathrm{id}_{M^\vee})(\mathrm{id}_P \otimes_k \mathrm{db}_M).$$

From the canonical isomorphism $\mathrm{Hom}_R(P \otimes_k M, N) \cong \mathrm{Hom}_k(P, \mathrm{Hom}_R(M, N))$, we deduce a functorial isomorphism

$$N \otimes_R M^\vee \cong \mathrm{Hom}_R(M, N).$$

Since the functor $- \otimes_R M^\vee$ is right exact, M is projective. Setting $N = R$ in the isomorphism above we obtain isomorphism $M^\vee \cong \mathrm{Hom}_R(M, R) = M^*$, by means of which the map ev is given by $\mathrm{ev}(\varphi \otimes m) = \varphi(m)$. For $N = M$, the identity map id_M corresponds to the element $\mathrm{db}(1) = \sum_{i=1}^{d} m_i \otimes \varphi^i$, with the property $m = \sum_{i=1}^{d} m_i \varphi^i(m)$, for all $m \in M$. Hence $\{m_i\}$ generate M

and $\{\varphi^i\}$ generate M^*. We call the pair $\{m_i\}$, $\{\varphi^i\}$ *dual bases* with respect to $\mathrm{db} = \mathrm{db}_{k,M}$. In particular we have proved

LEMMA 1.3.1: *Let M be an f.g. projective right R-module. Denote the action of R on M by τ and the one on M^* by σ. Then $M^* \otimes_k M$ is an R-bimodule and $\mathrm{ev}_{k,M}$ is an R-bimodule homomorphism.*

Assume now that M is an R-bimodule which is f.g. projective as an R-module. The the left action R on M induces a right action of R on $M^* = \mathrm{Hom}_R(M, R)$:

$$(\varphi r)(m) := \varphi(rm).$$

For all $r \in R$, $m \in M$, we have

$$\sum_i rm_i \varphi^i(m) = rm = \sum_i m_i \varphi^i(rm) = \sum_i m_i (\varphi^i r)(m),$$

hence

(1.10)
$$\sum_i rm_i \otimes \varphi^i = \sum_i m_i \otimes \varphi^i r.$$

Therefore, the map $\mathrm{db}_{k,M}$ extends to a map $\mathrm{db}_M : R \to M \otimes_R M^*$ of R-bimodules. On the other hand, we also have an R-bimodule map $\mathrm{ev}_M : M^* \otimes_R M \to R$, $\varphi \otimes m \mapsto \varphi(m)$, since $\varphi(rm) = (\varphi r)(m)$. It is easy to check the following identities for ev_M and db_M:
(1.11)
$$(\mathrm{ev}_M \otimes_R \mathrm{id}_{M^*})(\mathrm{id}_{M^*} \otimes_R \mathrm{db}_M) = \mathrm{id}_{M^*}; \quad (\mathrm{id}_M \otimes_R \mathrm{ev}_M)(\mathrm{db}_M \otimes_R \mathrm{id}_M) = \mathrm{id}_M.$$

In the language of monoidal categories we call such an R-bimodule M a left rigid object (in R-Bimod) and M^* the left dual to M.

We also have the notion of right dual to a left R-module as well as the notion of right rigid R-bimodules. In particular, the dual bimodule M^* to M, if it exists, is right rigid and the right dual to M^* is M.

We define now the stucture of an R-coring on $M^* \otimes_k M$ for a finitely generated projective right R-module M. Denote by τ the action of R on $M^* \otimes_k M$ which is given by the action of R on M and denote by σ the action on $M^* \otimes_k M$ which is given by the action on M^*. Set

$$\Delta := \mathrm{id}_{M^*} \otimes_k \mathrm{db}_{k,M} \otimes_k \mathrm{id}_{M^*} : M^* \otimes_k M \to M^* \otimes_k M \otimes_R M^* \otimes_k M,$$

and $\varepsilon := \mathrm{ev}_{k,M}$. It follows immediately from (1.9) that $M^* \otimes_k M$ is an R-coring. If, moreover, M is an R-bimodule, then there are four actions of R on $M^* \otimes_k M$. Thus we have proved

LEMMA 1.3.2: *Let M be an R-bimodule which is f.g. projective as a right R-module. Denote the actions of R on $M^* \otimes_k M$ induced from those on M by τ and the actions of R on $M^* \otimes_k M$ induced from those on M^* by σ. Then $(M^* \otimes_k M, \mathrm{ev}_{k,M})$ is an R-coalgebroid.*

In particular, $R \otimes_k R$ is an R-coalgebroid, the actions of R on $R \otimes_k R$ are specified as follows:

$$\sigma(a)\tau(b)(m \otimes n)\sigma(c)\tau(d) = amc \otimes bnd.$$

1.4. COMODULES OVER COALGEBROIDS. Let M be a right R-module and L be an R-coalgebroid. Denote by τ the right action of R on M. We form the tensor product $M_\tau \otimes^\sigma L$. On this module there are three actions of R, induced from its actions on L. A **coaction** of L on M is a map $\delta : M \longrightarrow M_\tau \otimes^\sigma L$, satisfying the following conditions:

$$\delta(m\tau(a)) = \delta(m)\tau(a) \quad \text{(the linearity on } R\text{),}$$
$$(\delta_\tau \otimes^\sigma \mathrm{id}_L)\delta = (\mathrm{id}_M {}_\tau \otimes^\sigma \Delta)\delta \quad \text{(the coassociativity),}$$
$$(\mathrm{id}_M {}_\tau \otimes^\sigma \varepsilon)\delta = \mathrm{id}_L \quad \text{(the unity).}$$

In other words, M is a comodule over the R-coring L with respect to the (σ, τ) action (R acts on the left by σ and on the right by τ). We use Sweedler's notation for the coaction $\delta(m) = \sum_{(m)} m_{(0)\,\tau} \otimes^\sigma m_{(1)}$. Analogously, for a left R-module M with the action denoted by σ, we can define the notion of a left coaction of L on M.

Example: For an R-bimodule M which is f.g. projective as right module, M is a right comodule over $L = M^* \otimes_k M$ and M^* is a left comodule over L. The action is given as follows: $\delta(m) = \sum_i m_i \otimes_R (\varphi_i \otimes_k m)$. In particular, for $M = R$, the coaction of $R \otimes_k R$ on R is given by $\delta(a) = 1 \otimes_R (1 \otimes_k a)$. Note that in this definition, we cannot move a to the left, i.e., $\delta(a) \neq a \otimes (1 \otimes 1)$, unless a is in k.

Let L be an R-coalgebroid and M a right comodule over L. We set

$$(1.12) \qquad \tau(a)m := \sum_{(m)} m_{(0)}\tau\varepsilon\left(\tau(a)m_{(1)}\right).$$

This definition does not depend on the choice of $m_{(0)}$ and $m_{(1)}$. Indeed, we have, for $m \in M$, $l \in L$, $a, b \in R$,

$$m\tau(b)\tau\varepsilon\left(\tau(a)l\right) = m\tau\varepsilon\left(\sigma(b)\tau(a)l\right) = m\tau\varepsilon\left(\tau(a)(\sigma(b)l)\right).$$

LEMMA 1.4.1: *Let L be an R-coalgebroid. Then the action defined in (1.12) makes M an R-bimodule. δ is R-linear with respect to this new action on M and the action on $M\ _\tau\otimes^\sigma L$ specified above, i.e.,*

$$\delta(\tau(a)m) = \sum_{(m)} m_{(0)} \otimes \tau(a)m_{(1)}.$$

Furthermore, δ satisfies the equation

$$(1.13) \qquad \sum_{(m)} \tau(a)m_{(0)} \otimes m_{(1)} = \sum_{(m)} m_{(0)} \otimes m_{(1)}\sigma(a).$$

Conversely, a left action on M of R with respect to which δ is linear in the above sense is uniquely given by the formula in (1.12).

The proof contains lengthy verifications using definitions and the relations in (1.3,1.4,1.5,1.13) and will be omitted.

Remark: By virtue of Lemma 1.4.1, by a (right) comodule over an R-algebroid L we shall understand an R-bimodule equipped with a coaction δ satisfying the conditions of this lemma. It is however not true that if M is a right L-comodule and N is an R-bimodule, then $N \otimes_R M$ is an L-comodule for this would contradict Lemma 1.4.1.

Analogously, we have a notion of left L-comodules. Sweedler's notation for $\delta : M \longrightarrow L\ _\tau\otimes^\sigma M$ reads $\delta(\varphi) = \sum_{(\varphi)} \varphi_{(-1)}\ _\tau\otimes^\sigma \varphi_{(0)}$. As in the case of right comodules, we can define a right action of R on a left L-comodule

$$(1.14) \qquad \varphi\sigma(a) := \sum_{(\varphi)} \sigma\varepsilon\left(\varphi_{(-1)}\sigma(a)\right)\varphi_{(0)}.$$

This action is well-defined and an analog of Lemma 1.4.1 holds:

$$(1.15) \qquad \sum_{(\varphi)} \tau(a)\varphi_{(-1)} \otimes \varphi_{(0)} = \sum_{(\varphi)} \varphi_{(-1)} \otimes \varphi_{(0)}\sigma(a).$$

LEMMA 1.4.2: *Let M be a right L-comodule which is f.g. projective as a right module over R. Then there is a left coaction of L on M^* given by the condition*

$$\sum_{(\varphi)} \varphi_{(-1)} \tau \varphi_{(0)}(m) = \sum_{(m)} \sigma \varphi(m_{(0)}) m_{(1)}.$$

This correspondence is one-to-one between right L-comodules, f.g. projective as right R-modules and left L-comodules, f.g. projective as left R-modules.

Proof. Given a right L-comodule M. The coaction on M^* is given by

$$
\begin{array}{ccc}
M^* & \xrightarrow{\ \ \delta\ \ } & L \,_\tau\!\otimes^\sigma M^* \\
{\scriptstyle \mathrm{id}\otimes \mathrm{db}_{k,M}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{ev}_{k,M}\otimes\mathrm{id}} \\
M^* \otimes_k M \,_\tau\!\otimes^\sigma M^* & \xrightarrow[\ \mathrm{id}\otimes\delta\otimes\mathrm{id}\ ]{} & M^* \otimes_k (M \,_\tau\!\otimes^\sigma L) \,_\tau\!\otimes^\sigma M^*.
\end{array}
$$

More explicitly, let m_i and φ^i, $i = 1, 2, \ldots, d$ be a pair of dual bases in M and M^* respectively. The coaction on M^* is given by

$$\delta(\varphi) = \sum_i \sigma \varphi(m_{i(0)}) m_{i(1)} \otimes \varphi^i.$$

The verification is straightforward.

Conversely, given a left coaction $\delta : M \longrightarrow L \,_\tau\!\otimes^\sigma M$, where M is an f.g. projective left R-module, one defines a right coaction of L on the right dual $^*M$ by the condition $\sum_{(m)} m_{(-1)} \tau \eta(m_{(0)}) = \sum_{(\eta)} \sigma \eta_{(0)}(m) \eta_{(1)}$. It is explicitly given by

$$
\begin{array}{ccc}
^*M & \xrightarrow{\ \ \delta\ \ } & ^*M \,_\tau\!\otimes^\sigma L \\
{\scriptstyle \mathrm{db}_{k,{}^*M}\otimes\mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}\otimes\mathrm{ev}_{k,{}^*M}} \\
^*M \,_\tau\!\otimes^\sigma M \otimes_k {}^*M & \xrightarrow[\ \mathrm{id}\otimes\delta\otimes\mathrm{id}\ ]{} & ^*M \,_\tau\!\otimes^\sigma (L \,_\tau\!\otimes^\sigma M) \otimes_k {}^*M \qquad \blacksquare
\end{array}
$$

1.5. TENSOR PRODUCTS OF COALGEBROIDS. R-coalgebroids form a category in a natural way: morphisms between two coalgebroids are those $R-R$-bimodules maps that commute with Δ and ε. In this section, we introduce a tensor product in this category. Let $\boxtimes$ denote the tensor product $^\sigma_\tau\!\otimes_\sigma^\tau$, which is given precisely by

$$L^\sigma_\tau\!\otimes^\tau_\sigma K = L \otimes_k K \Big/ \Big(\sigma(a)h\tau(b) \otimes k = h \otimes \tau(b)k\sigma(a) \Big),$$

for $R - R$-bimodules L and K. In other words, we have the following relation in $L \boxtimes K : \forall h \in L, k \in K$,

$$(1.16) \qquad\qquad \sigma(a)h\tau(b) \boxtimes k = h \boxtimes \tau(b)k\sigma(a).$$

We specify the following actions of R on $L \boxtimes K$:

$$\sigma(a)(h \boxtimes k)\sigma(b) = h\sigma(b) \boxtimes \sigma(a)k,$$
$$\tau(a)(h \boxtimes k)\tau(b) = \tau(a)h \boxtimes k\tau(b).$$

Here we adopt the convention that the action of R has preference over the tensor product.

Let L and K be R-coalgebroids. Define the k-linear maps

$$\bar{\Delta} : L \boxtimes K \longrightarrow (L \boxtimes K)\,_\tau \otimes^\sigma (L \boxtimes K) \quad \text{and} \quad \bar{\varepsilon} : L \boxtimes K \longrightarrow R$$

by

$$\bar{\Delta}(h \boxtimes k) = \sum_{(h),(k)} \left(h_{(1)} \boxtimes k_{(1)} \right)\,_\tau \otimes^\sigma \left(h_{(2)} \boxtimes k_{(2)} \right) \quad \text{and} \quad \bar{\varepsilon}(h \boxtimes k) = \varepsilon(k\sigma\varepsilon(h)).$$

The maps $\bar{\Delta}$ and $\bar{\varepsilon}$ are well-defined and they define a coalgebroid structure on $L \boxtimes K$ (cf. [20, 3.10]). Recall that $R \otimes_k R$ is an R-coalgebroid with the following $R - R$-bimodule structure

$$\sigma(a)\tau(b)(m \otimes n)\sigma(c)\tau(d) = amc \otimes bnd.$$

The category of R-coalgebroids is a monoidal category, with the unit object being $R \otimes_k R$.

1.6. R-BIALGEBROIDS. Since the category of R-coalgebroids is monoidal, we have the notion of monoids in this category, which are called R-bialgebroids. More explicitly, an R-bialgebroid L is a coalgebroid equipped with the following morphisms of $R - R$-bimodules $m : L \boxtimes L \longrightarrow L$ and $u : R \otimes_k R \longrightarrow L$, satisfying

$$(1.17) \qquad\qquad \Delta m = (m\,_\tau \otimes^\sigma m)\bar{\Delta}; \quad \varepsilon m = \bar{\varepsilon};$$

$$(1.18) \qquad\qquad \Delta u = u\,_\tau \otimes^\sigma u; \quad \varepsilon u(a \otimes b) = ab;$$

$$(1.19) \qquad\qquad m(\mathrm{id}_{R \otimes_k R} \boxtimes m) = m(m \boxtimes \mathrm{id}_{R \otimes_k R});$$

$$(1.20) \qquad\qquad m(\mathrm{id}_{R \otimes_k R} \boxtimes u) = m(u \boxtimes \mathrm{id}_{R \otimes_k R}) = \mathrm{id}_L,$$

where we use the identification $L \boxtimes (R \otimes_k R) \cong L \cong (R \otimes_k R) \boxtimes L$, which is given explicitly by

$$(1.21) \qquad\qquad h \boxtimes (a \otimes b) \longleftrightarrow \sigma(a)h\tau(b); \quad (a \otimes b) \boxtimes h \longleftrightarrow \tau(b)h\sigma(a).$$

Denoting $h \circ k = m(h \boxtimes k)$ and using Sweedler's notation, we have

$$\sigma(a)h\tau(b) \circ k = h \circ \tau(b)k\sigma(a), \quad \varepsilon(h \circ k) = \varepsilon(k\sigma(h))$$

(1.22)
$$\Delta(h \circ k) = \sum_{(h)(k)} h_{(1)} \circ k_{(1)} \,_\tau\otimes^\sigma h_{(2)} \circ k_{(2)}; \quad \Delta(1_L) = 1_L \,_\tau\otimes^\sigma 1_L,$$

bearing in mind the preference of $\circ$ over $\otimes$, where $1_L := u(1_R \otimes_k 1_R)$. On the other hand, the linearity of m and u can be expressed as

(1.23)
$$\sigma(a)\tau(b)(h \circ k)\tau(c)\sigma(d) = \tau(b)h\sigma(d) \circ \sigma(a)k\tau(c),$$

where we adopt the convention that the action of R has preference over the product.

We define the maps s and t from R to L as follows

$$s(a) = \sigma(a)1_L = 1_L\sigma(a); \quad t(a) = \tau(a)1_L = 1_L\tau(a).$$

The relations below follow immediately from (1.20) (or from (1.23))

(1.24)
$$\begin{aligned} s(a) \circ h = h\sigma(a); \quad h \circ s(a) = \sigma(a)h; \\ h \circ t(a) = h\tau(a); \quad t(a) \circ h = \tau(a)h. \end{aligned}$$

In particular, s is an anti-homomorphism and t is a homomorphism of k-algebras from $R \longrightarrow L$.

Bialgebroids over associative algebras seem to be first introduced by Takeuchi [19] and later independently introduced by J. Lu [7].

1.7. COMODULES OVER BIALGEBROIDS. A comodule over a bialgebroid is by definition a comodule over the underlying coalgebroid. We have seen in Subsection 1.4 that a right comodule over an R-coalgebroid, which is initially a right R-module, can be endowed with a structure of left R-module. In this subsection we show that the tensor product of two comodules over a bialgebroid is again a comodule.

Let M, N be right comodules over an R-bialgebroid L. Define a coaction of L on $M \otimes_R N$

$$\delta(m \otimes n) = \sum_{(m)(n)} m_{(0)} \otimes n_{(0)} \otimes m_{(1)} \circ n_{(1)}.$$

LEMMA 1.7.1: *The coaction given above is well-defined and makes $M \otimes_R N$ a comodule over L.*

Notice that R itself is a comodule over L by means of the morphism t defined in Subsection 1.6: $\delta(a) = 1 \otimes t(a) = 1 \otimes \tau(a)1_H$.

COROLLARY 1.7.2: *The category of comodules over a bialgebroid is monoidal with the unit object being R.*

1.8. THE ANTIPODE. Consider the tensor product $H\,{}^{\sigma}\!\otimes_{\sigma} H$ defined as follows

$$H\,{}^{\sigma}\!\otimes_{\sigma} H := H \otimes_k H \Big/ \Big(\sigma(a)h \otimes k = h \otimes k\sigma(a) \Big)$$

and specify the actions of R as follows

$$\tau(a)(h \otimes k)\tau(b) = \tau(a)h \otimes k\tau(b); \quad \sigma(a)(h \otimes k)\sigma(b) = h\sigma(b) \otimes \sigma(a)k.$$

There is an $R - R$-bimodule morphism $\pi : H\,{}^{\sigma}\!\otimes_{\sigma} H \longrightarrow H \boxtimes H$, which is a quotient map.

Definition (cf. [17]): Let H be an R-bialgebroid. An antipode on H is by definition a map $\nabla : H \longrightarrow H\,{}^{\sigma}\!\otimes_{\sigma} H; \nabla(h) = \sum_{(h)} h^- \,{}^{\sigma}\!\otimes_{\sigma} h^+$, satisfying the following conditions:

$$(1.25) \qquad \sum_{(h)} h^- \circ h^+{}_{(1)}\,{}_{\tau}\!\otimes^{\sigma} h^+{}_{(2)} = 1\,{}_{\tau}\!\otimes^{\sigma} h$$

$$(1.26) \qquad \sum_{(h)} h_{(1)} \circ h_{(2)}{}^-\,{}^{\sigma}\!\otimes_{\sigma} h_{(2)}{}^+ = 1\,{}^{\sigma}\!\otimes_{\sigma} h.$$

If such an antipode exists, H is called a Hopf algebroid.

Define a map $\beta : H\,{}^{\sigma}\!\otimes_{\sigma} H \longrightarrow H\,{}_{\tau}\!\otimes^{\sigma} H$ to be

$$(1.27)$$

$$
\begin{array}{ccc}
H\,{}^{\sigma}\!\otimes_{\sigma} H & \xrightarrow{\quad\beta\quad} & H\,{}_{\tau}\!\otimes^{\sigma} H \\
{\scriptstyle \mathrm{id}\otimes\Delta}\Big\downarrow & & \Big\uparrow{\scriptstyle m\otimes\mathrm{id}} \\
H\,{}^{\sigma}\!\otimes_{\sigma} (H\,{}_{\tau}\!\otimes^{\sigma} H) \xrightarrow{\;\cong\;} (H\,{}^{\sigma}\!\otimes_{\sigma} H)\,{}_{\tau}\!\otimes^{\sigma} H & \xrightarrow{\;\pi\;} & (H \boxtimes H)\,{}_{\tau}\!\otimes^{\sigma} H
\end{array}
$$

$$\beta\left(h\,{}^{\sigma}\!\otimes_{\sigma} k\right) = \sum_{(k)} h \circ k_{(1)}\,{}_{\tau}\!\otimes^{\sigma} k_{(2)}.$$

Then we have

$$\beta\left(\sum_{(k)} h \circ k^-\,{}^{\sigma}\!\otimes_{\sigma} k^+\right) = \sum_{(k)} h \circ k^- \circ k^+{}_{(1)}\,{}_{\tau}\!\otimes^{\sigma} k^+{}_{(2)} = h\,{}_{\tau}\!\otimes^{\sigma} k \text{ by (1.25)}$$

and

$$\sum_{(k)} h \circ k_{(1)} \circ k_{(2)}{}^-\,{}^{\sigma}\!\otimes_{\sigma} k_{(2)}{}^+ = h\,{}_{\tau}\!\otimes^{\sigma} k \text{ by (1.26) .}$$

Therefore the map β is invertible with the inverse given by

$$(1.28) \qquad \beta^{-1}(h\,_\tau\otimes^\sigma k) = \sum_{(k)} h \circ k^-\,{}^\sigma\otimes_\sigma k^+.$$

We have $\nabla(h) = \beta^{-1}(1 \otimes h)$, whence ∇ is uniquely determined. Thus, if an antipode exists, then it is determined uniquely.

Remark: If $R = k$ and H is a Hopf algebra over k, then ∇ is given explicitly by $\nabla(h) = \sum_{(h)} S(h_{(1)}) \otimes h_{(2)}$ where S denotes the antipode of H.

LEMMA 1.8.1 ([17, Proposition 3.7]): *Let H be a Hopf algebroid. Then the antipode ∇ satisfies the following relations:*

$$(1.29) \qquad \nabla(1_H) = 1_H \otimes 1_H,$$

$$(1.30) \qquad \nabla(\tau(a)\sigma(b)h\sigma(c)\tau(d)) = \sum_{(h)} \tau(b)h^-\tau(c) \otimes \tau(a)h^+\tau(d),$$

$$(1.31) \qquad \sum_{(h)} h_{(1)}{}^-\,{}^\sigma\otimes_\sigma h_{(1)}{}^+\,{}_\tau\otimes^\sigma h_{(2)} = \sum_{(h)} h^-\,{}^\sigma\otimes_\sigma h^+{}_{(1)}\,{}_\tau\otimes^\sigma h^+{}_{(2)},$$

$$(1.32) \qquad \sum_{(h)} h^-\,{}^\sigma\otimes_\tau h^{+-}\,{}^\sigma\otimes_\sigma h^{++} = \sum_{(h)} h^-{}_{(2)}\,{}^\sigma\otimes_\tau h^-{}_{(1)}\,{}^\sigma\otimes_\sigma h^+,$$

$$(1.33) \qquad \sum_{(h)} h^+\sigma(\varepsilon(h^-)) = h,$$

$$(1.34) \qquad \sum_{(h)} h^- \circ h^+ = 1_H\tau\varepsilon(h).$$

Proof. Applying β on both sides of equations (1.29) and (1.30) we obtain the identity maps. Thus, the equalities follows from the invertibility of β.

Applying β on the first two tensor components of both sides of (1.31) and using (1.25), we obtain the same values. Thus the equality also follows from the invertibility of β.

We prove (1.32). Let $\bar\beta$ be the map $H\,{}^\sigma\otimes_\tau H\,{}^\sigma\otimes_\sigma H \longrightarrow H\,{}_\tau\otimes^\sigma H\,{}_\tau\otimes^\sigma H$;

$$\bar\beta(h\,{}^\sigma\otimes_\tau k\,{}^\sigma\otimes_\sigma l) = k \circ l_{(1)}\,{}_\tau\otimes^\sigma h \circ l_{(2)}\,{}_\tau\otimes^\sigma l_{(3)}$$

$\bar\beta$ is also invertible with the inverse given by

$$\bar\beta^{-1}(k\,{}_\tau\otimes^\sigma h\,{}_\tau\otimes^\sigma l) = h \circ l^-\,{}^\sigma\otimes_\tau k \circ l^{+-}\,{}^\sigma\otimes_\sigma l^{++}.$$

Applying $\bar{\beta}$ to the right-hand side of (1.32) we obtain:

$$\sum_{(h)} h^-{}_{(1)} \circ h^+{}_{(1)} \; {}_\tau\otimes^\sigma \; h^-{}_{(2)} \circ h^+{}_{(2)} \; {}_\tau\otimes^\sigma \; h^+{}_{(3)}$$

$$= (h^- \circ h^+{}_{(1)})_{(1)} \; {}_\tau\otimes^\sigma \; (h^- \circ h^+{}_{(1)})_{(2)} \; {}_\tau\otimes^\sigma \; h^+{}_{(2)} \quad \text{by (1.22)}$$

$$= 1 \; {}_\tau\otimes^\sigma \; 1 \; {}_\tau\otimes^\sigma \; h.$$

On the other hand, applying $\bar{\beta}$ to the left-hand side of (1.32), we obtain

$$\sum_{(h)} (h^+)^- \circ (h^+)^+{}_{(1)} \; {}_\tau\otimes^\sigma \; h^- \circ (h^+)^+{}_{(2)} \; {}_\tau\otimes^\sigma \; (h^+)^+{}_{(3)}$$

$$= \sum_{(h)} 1 \; {}_\tau\otimes^\sigma \; h^- \circ h^+{}_{(1)} \; {}_\tau\otimes^\sigma \; h^+{}_{(2)} \quad \text{by (1.25)}$$

$$= 1 \; {}_\tau\otimes^\sigma \; 1 \; {}_\tau\otimes^\sigma \; h.$$

The invertibility of $\bar{\beta}$ implies the equality in (1.32).

(1.33) follows from (1.26). Indeed, we have

$$h = \sum_{(h)} h_{(2)}{}^+ \sigma\varepsilon(h_{(1)} \circ h_{(2)}{}^-) = h_{(2)}{}^+ \sigma\varepsilon\left(h_{(2)}{}^- \sigma\varepsilon(h_{(1)})\right)$$

$$= h_{(2)}{}^+ \sigma\varepsilon\left(\tau\varepsilon(h_{(1)} h_{(2)}{}^-)\right) \quad \text{by condition (iii) for } \varepsilon$$

$$= h^+ \sigma\varepsilon(h^-) \quad \text{by (1.30) for } \tau(b) \text{ and by (1.2).}$$

(1.34) also follows immediately from (1.25) and (1.2). $\blacksquare$

PROPOSITION 1.8.2: *Let H be an R-Hopf algebroid and M a right H-comodule. Assume that M is an f.g. projective right R-module. Then there exists a coaction of H on M^* making it the dual object to M in the category of right H-comodules.*

Proof. We first define a coaction of H on M^*. The right coaction of H on M induces the left coaction of H on M^*: $\partial(\varphi) = \sum_{(\varphi)} \varphi_{(-1)} \; {}_\tau\otimes^\sigma \; \varphi_{(0)}$ by the condition

$$(1.35) \qquad \sum_{(\varphi)} \varphi_{(-1)} \tau\varphi_{(0)}(m) = \sum_{(m)} \sigma\varphi(m_{(0)}) m_{(1)}.$$

Define a right coaction of H on M^* as follows

$$(1.36) \qquad \delta(\varphi) := \sum_{(\varphi)} \sigma\varepsilon\left(\varphi_{(-1)}{}^+\right) \varphi_{(0)} \; {}_\sigma\otimes^\sigma \; \varphi_{(-1)}{}^-.$$

We will show that this is a well-defined coaction of H and that with this coaction M^* is a dual H-comodule to M.

It is easy to check that this coaction is well-defined, i.e., it does not depend on the choice of $\varphi_{(0)}, \varphi_{(-1)}$ and $\varphi_{(-1)}{}^-, \varphi_{(-1)}{}^+$. We show that δ is a coaction. For simplicity we shall use the notation $\phi := \varphi_{(-1)}$. Notice that

$$\sum_{(\varphi)} \varphi_{(-2)} \, {}_\tau\otimes^\sigma \, \varphi_{(-1)} \, {}_\tau\otimes^\sigma \, \varphi_{(0)} = \sum_{(\varphi)(\phi)} \phi_{(1)} \, {}_\tau\otimes^\sigma \, \phi_{(2)} \, {}_\tau\otimes^\sigma \, \varphi_{(0)}.$$

The coassociativity of δ amounts to the following equation

$$\sum_{(\varphi)} \sigma\varepsilon \left(\phi_{(2)}{}^+\right) \varphi_{(0)} \, {}_\sigma\otimes^\sigma \, \tau\varepsilon \left(\phi_{(1)}{}^+\right) \phi_{(2)}{}^- \, {}_\tau\otimes^\sigma \, \phi_{(1)}{}^-$$

$$= \sum_{(\varphi)} \sigma\varepsilon \left(\phi^+\right) \varphi_{(0)} \, {}_\sigma\otimes^\sigma \, \left(\phi^-\right)_{(1)} \, {}_\tau\otimes^\sigma \, \left(\phi^-\right)_{(2)}.$$

Applying ∇ on the last term of (1.31) and taking in account (1.30), we obtain the equality

$$\sum_{(h)} h_{(1)}{}^- \, {}^\sigma\otimes_\sigma \, h_{(1)}{}^+ \, {}_\tau\otimes^\tau \, h_{(2)}{}^- \, {}^\sigma\otimes_\sigma \, h_{(2)}{}^+$$

$$= \sum_{(h)} h^- \, {}^\sigma\otimes_\sigma \, h^+{}_{(1)} \, {}_\tau\otimes^\tau \, (h^+{}_{(2)})^- \, {}^\sigma\otimes_\sigma \, (h^+{}_{(2)})^+.$$

Therefore, for $h = \phi = \varphi_{(-1)}$, we have

$$\sum_{(\varphi)} \sigma\varepsilon \left(\phi_{(2)}{}^+\right) \varphi_{(0)} \, {}_\sigma\otimes^\sigma \, \tau\varepsilon \left(\phi_{(1)}{}^+\right) \phi_{(2)}{}^- \, {}_\tau\otimes^\sigma \, \phi_{(1)}{}^-$$

$$= \sum_{(\varphi)} \sigma\varepsilon \left(\left(\phi^+{}_{(2)}\right)^+ \right) \varphi_{(0)} \, {}_\sigma\otimes^\sigma \, \tau\varepsilon \left(\phi^+{}_{(1)}\right) \left(\phi^+{}_{(2)}\right)^- \, {}_\tau\otimes^\sigma \, \phi^-$$

$$= \sum_{(\varphi)} \sigma\varepsilon \left(\left(\sigma\varepsilon \left(\phi^+{}_{(1)}\right) \phi^+{}_{(2)} \right)^+ \right) \varphi_{(0)}$$

$$\quad {}_\sigma\otimes^\sigma \left(\sigma\varepsilon \left(\phi^+{}_{(1)}\right) \phi^+{}_{(2)} \right)^- \, {}_\tau\otimes^\sigma \, \phi^- \quad \text{(by (1.30))}$$

$$= \sum_{(\varphi)} \sigma\varepsilon \left((\phi^+)^+\right) \varphi_{(0)} \, {}_\sigma\otimes^\sigma \, (\phi^+)^- \, {}_\tau\otimes^\sigma \, \phi^- \quad \text{(by (1.2))}$$

$$= \sum_{(\varphi)} \sigma\varepsilon \left(\phi^+\right) \varphi_{(0)} \, {}_\sigma\otimes^\sigma \, \left(\phi^-\right)_{(1)} \, {}_\tau\otimes^\sigma \, \left(\phi^-\right)_{(2)} \quad \text{(by (1.32))}.$$

The counity amounts to the following equation

$$\varphi = \sum_{(\varphi)} \sigma\varepsilon(\phi^+)\varphi_{(0)}\sigma\varepsilon(\phi^-).$$

Indeed, by (1.14)

$$\sum_{(\varphi)} \left(\sigma\varepsilon\left(\phi^+\right)\varphi_{(0)}\right)\sigma\varepsilon(\phi^-) = \sum_{(\varphi)} \sigma\varepsilon\left[\left(\sigma\varepsilon\left(\phi_{(1)}^+\right)\phi_{(2)}\right)\sigma\varepsilon\left(\phi_{(1)}^-\right)\right]\varphi_{(0)}$$

$$= \sum_{(\varphi)} \sigma\varepsilon\left[\left(\sigma\varepsilon\left(\phi^+_{(1)}\right)\phi^+_{(2)}\right)\sigma\varepsilon\left(\phi^-\right)\right]\varphi_{(0)} \text{ by (1.31)}$$

$$= \sum_{(\varphi)} \sigma\varepsilon\left(\phi^+\sigma\varepsilon\left(\phi^-\right)\right)\varphi_{(0)} \quad \text{by (1.33)}$$

$$= \sum_{(\varphi)} \sigma\varepsilon\left(\varphi_{(-1)}\right)\varphi_{(0)} = \varphi.$$

Similar computation shows that the left action of R on M^*, which is induced from the right coaction of H as in (1.12) is just the natural one:

$$(a, \varphi) \longmapsto \sigma(a)\varphi : [\sigma(a)\varphi](m) = a\varphi(m).$$

Thus, Lemma 1.4.1 applies and the equation in (1.13) has the form

$$(1.37) \qquad \sigma(a)\sigma\varepsilon(\phi^+)\varphi_{(0)} {}_\sigma\otimes^\sigma \phi^- = \sigma\varepsilon(\phi^+)\varphi_{(0)} {}_\sigma\otimes^\sigma \phi^-\sigma(a)$$

Finally, we show that M^* equipped with this coaction is a left dual comodule to M, which amounts to showing that

$$\text{ev} : M^* {}_\sigma\otimes^\tau M \longrightarrow R \quad \text{and} \quad \text{db} : R \longrightarrow M {}_\tau\otimes^\sigma M^*$$

are morphisms of H-comodules. Choose a pair of dual bases on M and M^*, $\{m_i\}$ and $\{\varphi^i\}$, and denote for simplicity $\phi := \varphi_{(-1)}$ and $\phi^i := \varphi^i{}_{(-1)}$. We have to check the following equations:

$$(1.38) \qquad \sum_{(\varphi)} \varepsilon\left(\phi^+\right)\varphi_{(0)}(m_{(0)}) {}_\tau\otimes^\sigma \left(\phi^- \circ m_{(1)}\right) = 1 {}_\tau\otimes^\sigma 1\tau\varphi(m),$$

$$(1.39) \quad \sum_{i,(\varphi^i)} m_{i(0)} {}_\tau\otimes^\sigma \sigma\varepsilon\left(\phi^{i+}\right)\varphi^i{}_{(0)} {}_\tau\otimes^\sigma \left(m_{i(1)} \circ \phi^{i-}\right) = \sum_i m_i {}_\tau\otimes^\sigma \varphi^i {}_\tau\otimes^\sigma 1_H$$

Notice that (1.39) is equivalent to the following: for all $\varphi \in M^*$,

$$(1.40) \qquad \sum_{i,(\varphi^i)} \sigma\varphi(m_{i(0)})\sigma\varepsilon\left(\phi^{i+}\right)\varphi^i{}_{(0)} {}_\tau\otimes^\sigma \left(m_{i(1)} \circ \phi^{i-}\right) = \varphi {}_\tau\otimes^\sigma 1_H.$$

We prove (1.38):

$$\sum_{(\varphi)} \varepsilon\left(\phi^+\right)\varphi_{(0)}(m_{(0)})\,_\tau\otimes^\sigma\left(\phi^-\circ m_{(1)}\right)$$

$$=\sum_{(\varphi)}\varepsilon\left(\phi^+\right)\,_\tau\otimes^\sigma\left(\phi^-\circ\sigma\varphi_{(0)}(m)m_{(1)}\right)\quad\text{by (1.23)}$$

$$=\sum_{(\varphi)}\varepsilon\left(\phi_{(1)}{}^+\right)\,_\tau\otimes^\sigma\left(\phi_{(1)}{}^-\circ\phi_{(2)}\tau\varphi_{(0)}(m)\right)\quad\text{by (1.35)}$$

$$=\sum_{(\varphi)}\varepsilon\left(\phi^+{}_{(1)}\right)\,_\tau\otimes^\sigma\left(\phi^-\circ\phi^+{}_{(2)}\tau\varphi_{(0)}(m)\right)\quad\text{by (1.31)}$$

$$=\sum_{(\varphi)}1\,_\tau\otimes^\sigma\left(\phi^-\circ\left(\sigma\varepsilon\left(\phi^+{}_{(1)}\right)\phi^+{}_{(2)}\right)\right)\tau\varphi_{(0)}(m)\quad\text{by (1.23)}$$

$$=\sum_{(\varphi)}1\,_\tau\otimes^\sigma\left(\phi^-\circ\phi^+\right)\tau\varphi_0(m)\quad\text{by (1.34)}$$

$$=1\,_\tau\otimes^\sigma 1\tau\varphi(m).$$

For (1.40), we first notice that, on applying δ on both sides of the equations $\varphi=\sum_i\sigma\varphi(m_i)\varphi^i$, we have

$$(1.41)\qquad\sum_{(\varphi)}\sigma\varepsilon\left(\phi^+\right)\varphi_{(0)}\otimes\phi^-=\sum_{i,(\varphi^i)}\sigma\varepsilon\left(\phi^{i+}\right)\varphi^i{}_{(0)}\otimes\tau\varphi(m_i)\phi^{i-}.$$

Now, the left hand side of (1.40) is equal to

$$\sum_{i,(\varphi)}\sigma\varepsilon\left(\phi^{i+}\right)\varphi^i{}_{(0)}\,_\tau\otimes^\sigma\left(m_{i(1)}\circ\phi^{i-}\sigma\varphi(m_{i(0)})\right)\quad\text{by (1.37)}$$

$$=\sum_{i,(\varphi)}\sigma\varepsilon\left(\phi^{i+}\right)\varphi^i{}_{(0)}\,_\tau\otimes^\sigma\left(\sigma\varphi(m_{i(0)})m_{i(1)}\circ\phi^{i-}\right)\quad\text{by (1.22)}$$

$$=\sum_{i,(\varphi)}\sigma\varepsilon\left(\phi^{i+}\right)\varphi^i{}_{(0)}\,_\tau\otimes^\sigma\left(\phi\tau\varphi_{(0)}(m_i)\circ\phi^{i-}\right)\quad\text{by (1.35)}$$

$$=\sum_{i,(\varphi)}\sigma\varepsilon\left(\phi^{i+}\right)\varphi^i{}_{(0)}\,_\tau\otimes^\sigma\left(\phi\circ\tau\varphi_{(0)}(m_i)\phi^{i-}\right)$$

$$=\sum_{i,(\varphi)}\sigma\varepsilon\left(\phi^{i+}\right)\varphi^i{}_{(0)}\,_\tau\otimes^\sigma\left(\phi_{(1)}\circ\phi^-\right)\quad\text{by (1.41)}$$

$$=\sum_{(\varphi)}\sigma\varepsilon\left(\phi\right)\varphi_{(0)}\,_\tau\otimes^\sigma 1_H\quad\text{by (1.41)}$$

$$=\varphi\,_\tau\otimes^\sigma 1_H.$$

The proof is complete. ∎

1.9. THE OPPOSITE ANTIPODE. We have seen that for a Hopf algebroid, each comodule which is f.g. projective over R possesses a left dual. As we know in the case of Hopf algebras, a right dual can be defined in terms of the inverse to the antipode, i.e., if the antipode is bijective, each finite dimensional comodule possesses a right dual. A (sufficient) condition for the existence of the right dual to a f.g. projective comodule of a Hopf algebroid can be expressed as the existence of a map generalizing the map $h \longmapsto S^{-1}(h_{(2)}) \otimes h_{(1)}$ for Hopf algebras (see Remark in Subsection 1.8).

Definition: Let H be a bialgebroid. An **opposite antipode** is a map $\nabla^{\mathrm{op}} : H \longrightarrow H \,_\tau \otimes^\tau H$, $\nabla^{\mathrm{op}}(h) := \sum_{(h)} h_- \,_\tau \otimes^\tau h_+$, satisfying the following axioms:

$$\sum_{(h)} h_{+(1)} \,_\tau \otimes^\sigma h_- \circ h_{+(2)} = h \,_\tau \otimes^\sigma 1$$

$$\sum_{(h)} h_{(2)} \circ h_{(1)-} \,_\tau \otimes^\tau h_{(1)+} = 1 \,_\tau \otimes^\tau h.$$

LEMMA 1.9.1: *Let H be a bialgebroid with an opposite antipode ∇^{op}. Define a map $\gamma : H \,_\tau \otimes^\tau H \longrightarrow H \,_\tau \otimes^\sigma H$, $\gamma(h \otimes k) = \sum k_{(1)} \,_\tau \otimes^\sigma h \circ k_{(2)}$. Then γ is invertible with the inverse given by $\gamma^{-1}(h \,_\tau \otimes^\sigma k) = \sum k \circ h_- \,_\tau \otimes^\tau h_+$. Further the map ∇^{op} satisfies the following equations:*

$$\nabla^{\mathrm{op}}(\tau(a)\sigma(b)h\sigma(c)\tau(d) = \sum_{(h)} \sigma(a)h_-\sigma(d) \,_\tau \otimes^\tau \sigma(b)h_+\sigma(c)$$

$$\sum_{(h)} h_{+(1)} \,_\tau \otimes^\sigma h_{+(2)} \,^\tau \otimes_\tau h_- = \sum_{(h)} h_{(1)} \,_\tau \otimes^\sigma h_{(2)+} \,^\tau \otimes_\tau h_{(2)-}$$

$$\sum_{(h)} h_+ \,_\tau \otimes^\sigma h_{+-} \,_\tau \otimes^\tau h_{++} = \sum_{(h)} h_{-(1)} \,_\tau \otimes^\sigma h_{-(2)} \,_\tau \otimes^\tau h_+.$$

Consequently, the opposite antipode is determined uniquely.

We call a bialgebroid equipped with an opposite antipode **opposite Hopf algebroid**.

PROPOSITION 1.9.2: *Let H be an opposite R-Hopf algebra and*

$$\delta : M \longrightarrow M \,_\tau \otimes^\sigma H$$

a right coaction of H on M. Then the opposite antipode induces a left coaction of H on M, $M \longrightarrow H_{\tau}\otimes^{\tau} M$, given by

$$m \longmapsto \sum m_{(1)-} {}_{\tau}\otimes^{\tau} m_{(0)}\tau\varepsilon(m_{(1)+}).$$

Assume that M is an f.g. projective left R-module with the right dual ${}^*M$. Then there exists a coaction $\rho : {}^*M \longrightarrow {}^*M_{\sigma}\otimes^{\sigma} H$, $\eta \longmapsto \sum_{(\eta)} \eta_{(0)} \otimes \eta_{(1)}$, making it a right dual H-comodule to M. ρ is given by the following condition

$$\sum_{(m)(\eta)} m_{(1)-}\tau\eta(m_{(0)}\tau\varepsilon(m_{(1)+})) = \sum_{(m)(\eta)} \sigma\eta_{(0)} \left(m_{(0)}\tau\varepsilon(m_{(1)})\right) \eta_{(1)}.$$

The proof of these facts is left to the reader.

2. The Tannaka-Krein duality

2.1. TANNAKA-KREIN DUALITY FOR CORINGS. We fix as in Section 1 a commutative ring k and assume that everything is k-linear. Let R be a k-algebra. The Tannaka-Krein duality for R-corings was proved by P. Deligne [2]. Our presentation here follows A. Bruguières [1].

Let $\mathfrak{C}$ be a category and $\mathcal{F} : \mathfrak{C} \longrightarrow \mathsf{Mod}\text{-}R$ be a functor to the category of right R-modules. We define the Coend of $\mathcal{F}$ to be an R-bimodule L satisfying the following universal property: for any R-bimodule C, there is a natural isomorphism

(2.1) $$\mathrm{Nat}_R(\mathcal{F}, \mathcal{F} \otimes_R C) \cong {}_R\mathrm{Hom}_R(L, C).$$

Here we use the convention of Subsection 1.1 for the Hom. By the universal property, L, if it exists, is uniquely determined up to an isomorphism.

If the image of $\mathcal{F}$ lies in the subcategory of right f.g. projective R-modules, L can be constructed as follows. First notice that for any R-bimodule C and any object $X \in \mathfrak{C}$, we have by means of the projectivity of $\mathcal{F}(X)$

$$\mathrm{Hom}_R(\mathcal{F}(X), \mathcal{F}(X) \otimes_R C) \cong {}_R\mathrm{Hom}_R(\mathcal{F}(X)^* \otimes_k \mathcal{F}(X), C).$$

Thus we form the direct sum

(2.2) $$L_0 := \bigoplus_{X \in \mathfrak{C}} \mathcal{F}(X)^* \otimes_k \mathcal{F}(X)$$

and for any morphism $f : X \to Y$ in $\mathfrak{C}$, consider the (inner) diagram of R-bimodule maps

(2.3)

$$\mathcal{F}(Y)^* \otimes_k \mathcal{F}(X) \xrightarrow{\mathcal{F}(f)^* \otimes \mathrm{id}} \mathcal{F}(X)^* \otimes_k \mathcal{F}(X)$$

with vertical map $\mathrm{id} \otimes \mathcal{F}(f)$ down to $\mathcal{F}(Y)^* \otimes_k \mathcal{F}(Y)$, a second vertical arrow down to L_0, and $\mathcal{F}(Y)^* \otimes_k \mathcal{F}(Y) \longrightarrow L_0$, with arrows converging to L.

Let L be the maximal quotient R-bimodule of L_0 which makes all the above (outer) diagrams commute. Then it is easy to see that the R-bimodule L satisfies the universal property in (2.1).

We will show that L is an R-coring in the sense of Subsection 1.1. As in Subsection 1.3 we denote the actions of R on $\mathcal{F}(X)$ by τ and the actions on its dual by σ. Thus $\mathcal{F}(X)^* \otimes_k \mathcal{F}(X)$ is an R-bimodule by means of the actions τ and σ. The actions of R on L_0 and L will be named in the same way.

Set $C = L$ in (2.1). Then the identity $L \to L$ corresponds through the isormophism in (2.1) to a natural transformation $\delta : \mathcal{F} \longrightarrow \mathcal{F} \otimes_R L$. For an arbitrary natural transformation $\rho : \mathcal{F} \longrightarrow \mathcal{F} \otimes_R C$, the naturality of (2.1) on C implies that the corresponding morphism $f_\rho : L \longrightarrow C$ satisfies

(2.4) $$\rho = (\mathrm{id} \otimes f_\rho)\delta.$$

For $C = L \,_\tau\otimes^\sigma L$ the natural morphism

$$(\delta \otimes \mathrm{id})\delta : \mathcal{F} \to \mathcal{F} \otimes_R L \,_\tau\otimes^\sigma L$$

corresponds though the isomorphism in (2.1) to a morphism $\Delta : L \longrightarrow L \,_\tau\otimes^\sigma L$, which according to (2.4) satisfies

$$(\delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \Delta)\delta.$$

For $C = R$, the identity transformation corresponds to a morphism: $\varepsilon : L \longrightarrow R$. It is easy to show that (L, Δ, ε) is an R-coring.

LEMMA 2.1.1: *Through the isomorphism in (2.1), if $\delta \in \mathrm{Nat}(\mathcal{F}, \mathcal{F} \otimes_R C)$ is a family of coactions of an R-coring C, then the corresponding morphism $L \longrightarrow C$ is a morphism of R-corings.*

Proof. Let $\varphi : L \longrightarrow C$ be the map that corresponds to δ. Since δ is a coaction of C on $\mathcal{F}(X)$ for every X, we have two equal maps $(\delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \Delta)\delta : \mathcal{F}(X) \longrightarrow \mathcal{F}(X) \otimes C \otimes C$. These maps should correspond to the same map $L \longrightarrow C \otimes C$, which means $\Delta_C \varphi = (\varphi \otimes \varphi)\Delta_L$. The commutativity of φ with the counits also follows from the universal property of L. ∎

Thus, given a category $\mathfrak{C}$ and a functor $\mathcal{F} : \mathfrak{C} \longrightarrow$ Mod-R with image in the subcategory of f.g. projective modules, then $\mathcal{F}$ factors through a functor $\bar{\mathcal{F}} : \mathfrak{C} \longrightarrow$ comod-L, and the forgetful functor. This is the first part of the Tannaka-Krein duality. The second part, which is usually more difficult, is to prove that $\bar{\mathcal{F}}$ is an equivalence if $\mathfrak{C}$ is a "good" abelian category.

From now on we shall assume that k is a field. Recall that a k-linear abelian category $\mathfrak{C}$ is said to be **locally finite** (over k) if each its Hom-set is finite dimension over k and each object has the composition series of finite length.

Definition: Let R be a k-algebra and L be an R-coring. L is said to be (right) semi-transitive if the following conditions are satisfied:

(i) each right L-comodule is projective over R.
(ii) each L-comodule is a filtered limit of subcomodule which is finitely generated over R.
(iii) the category comod-L of right L-comodules which are finitely generated as R-modules is locally finite over k.

THEOREM 2.1.2 ([2], see also [1, Theorem 5.2]): *Let k be a field and $\mathfrak{C}$ be a (small) k-linear abelian category which is locally finite. Let $\mathcal{F} : \mathfrak{C} \longrightarrow$ mod-R be an exact faithful functor with image in the subcategory of f.g. projective modules. Let $L =$ Coend$(\mathcal{F})$. Then L is a semi-transitive coring and the functor $\bar{\mathcal{F}}$ is an equivalence of abelian categories. Conversely, let L be a semi-transitive coring and $\mathcal{F} :$ comod-$L \longrightarrow$ mod-R be the forgetful functor. Then $\mathcal{F}$ is faithful, exact and has image in the category of projective modules of finite rank and $L \cong$ Coend$(\mathcal{F})$.*

2.2. TANNAKA-KREIN DUALITY FOR BIALGEBROIDS. Let $\mathfrak{C}$ be a k-linear category and $\mathcal{F} : \mathfrak{C} \longrightarrow R$-Bimod a functor with image in the subcategory of left rigid R-bimodules (i.e., f.g. projective as right R-modules). Then we can construct the Coend of $\mathcal{F}$, denoted by L. There are several actions of R on L which we will specify now.

Recall from Subsection 1.3 that the left dual $\mathcal{F}(X)^*$ to $\mathcal{F}(X)$ is also an R-bimodule. We shall use the convention of 1.3 for denoting the actions of R on $\mathcal{F}(X)^* \otimes_k \mathcal{F}(X)$. The actions of R on L_0 will be denoted accordingly. Since the maps $\mathcal{F}(f)^* \otimes \mathrm{id}$ and $\mathrm{id} \otimes \mathcal{F}(f)$ in the diagram (2.3) commute with all the (left and right) actions σ and τ, there are natural actions of R on L which will be denoted accordingly. As shown in Subsection 2.1, L with respect to the bimodule structure given by (σ, τ) is an R-coring.

LEMMA 2.2.1: *Let $\mathcal{F} : \mathfrak{C} \longrightarrow R$-Bimod be a functor with image in the subcategory of left rigid bimodules. Then $L = \mathsf{Coend}(\mathcal{F})$ is a coalgebroid.*

Proof. As shown in the previous subsection, L is an R-coring with respect to the actions (σ, τ). It remains to show that Δ is a morphism of $R - R$-bimodules and that ε satisfies $\varepsilon(\tau(r)a) = \varepsilon(a\sigma(r))$, $\forall r \in R, a \in L$.

To see that Δ is a morphism of $R - R$-bimodules, it is sufficient to notice that in the construction of L (diagrams in (2.3)) all maps are $R - R$-bimodules morphisms and for each object $X \in \mathfrak{C}$, the coproduct

$$\Delta_X : \mathcal{F}(X)^* \otimes_k \mathcal{F}(X) \longrightarrow \mathcal{F}(X)^* \otimes_k \mathcal{F}(X) \otimes_R \mathcal{F}(X)^* \otimes_k \mathcal{F}(X)$$

is a morphism of $R - R$-bimodules, for $\mathcal{F}(X)^* \otimes_k \mathcal{F}(X)$ is a coalgebroid (see Subsection 1.3).

Similarly, the counit $\varepsilon_X : \mathcal{F}(X)^* \otimes_k \mathcal{F}(X) \longrightarrow R$ satisfies $\varepsilon_X(\tau(r)a) = \varepsilon_X(a\sigma(r))$ and moreover, for any pair of objects $X, Y \in \mathfrak{C}$, a morphism $f : X \longrightarrow Y$ induces a morphism $\varepsilon_f : \mathcal{F}(Y)^* \otimes_k \mathcal{F}(X) \longrightarrow R$ which is linear with respect to the actions (σ, τ) and satisfies $\varepsilon_f(\tau(r)a) = \varepsilon_f(a\sigma(r))$. Therefore, we have commutative diagrams of the form

(2.5)

$$
\begin{array}{ccc}
\mathcal{F}(Y)^* \otimes_k \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f) \otimes \mathrm{id}} & \mathcal{F}(X)^* \otimes_k \mathcal{F}(X) \\
& \searrow^{\varepsilon_f} \quad \swarrow^{\varepsilon_X} & \\
{\scriptstyle \mathrm{id} \otimes \mathcal{F}(f)} \Big\downarrow & R & \Big\downarrow \\
& \nearrow^{\varepsilon_Y} \quad \nwarrow_{\varepsilon} & \\
\mathcal{F}(Y)^* \otimes_k \mathcal{F}(Y) & \longrightarrow & L \, .
\end{array}
$$

By construction, L is a quotient of L_0, which is the direct sum of $\mathcal{F}(X)^* \otimes_k \mathcal{F}(X)$, $X \in \mathfrak{C}$. We therefore conclude that the induced map $\varepsilon : L \to R$ also satisfies the equation $\varepsilon(\tau(r)a) = \varepsilon(a\sigma(r))$. Thus, L is an R-coalgebroid. ∎

LEMMA 2.2.2: *Let C be an R-coalgebroid. Let $\delta \in \mathrm{Nat}(\mathcal{F}, \mathcal{F} \otimes_R C)$ be a natural transformation, which is a family of coactions of a coalgebroid C satisfying equation (1.13). Then δ corresponds though the isomorphism in (2.1) to a morphism $L \longrightarrow C$ of coalgebroids.*

Proof. A coaction of a coalgebroid C on a left rigid bimodule $\mathcal{F}(X)$, which satisfies (1.13), induces a morphism of coalgebroids $\bar{\bar{\delta}} : \mathcal{F}(X)^* \otimes_k \mathcal{F}(X) \longrightarrow C$. Consequently the map

$$\sum_{X \in \mathfrak{C}} \bar{\delta}_X : L_0 = \bigoplus_{x \in \mathfrak{C}} \mathcal{F}(X)^* \otimes_k \mathcal{F}(X) \longrightarrow C$$

is also a homomorphism of R-coalgebroids. On the other hand, according to Lemma 2.1.1, there exists a homomorphism of R-corings $\varphi : L \longrightarrow C$ which fits in the following commutative diagrams for all $X \in \mathfrak{C}$

$$
\begin{array}{ccc}
L_0 & \longrightarrow & L \\
 & \searrow{\scriptstyle \sum_{X \in \mathfrak{C}} \bar{\delta}_X} & \downarrow{\scriptstyle \varphi} \\
 & & C.
\end{array}
$$

Since the map $L_0 \longrightarrow L$ is surjective, the R-linearity (with respect to all actions) of φ follows from the R-linearity of the maps $L_0 \longrightarrow L$ and $L_0 \longrightarrow C$. Thus φ is a homomorphism of R-coalgebroids. ∎

We recall that the tensor product $\boxtimes$ was introduced in Subsection 1.5.

PROPOSITION 2.2.3: *Let $\mathcal{F}$ and $\mathcal{G}$ be functors $\mathfrak{C} \longrightarrow R$-Bimod with images in the category of left rigid bimodules. Let $L = \mathsf{Coend}(\mathcal{F})$ and $K = \mathsf{Coend}(\mathcal{G})$. Then*

$$(2.6) \qquad \mathsf{Coend}(\mathcal{F} \otimes_R \mathcal{G}) \cong L \boxtimes K.$$

Proof. We still keep the notation for the actions of R on $M^* \otimes_k M$, $M \in R - R$-bimod, as in Subsection 1.3. We notice the following isomorphism for the $\boxtimes$-product

$$(2.7) \qquad (M \,_\tau\!\otimes^\tau N)^* \otimes_k (M \,_\tau\!\otimes^\tau N) \cong (M^* \otimes_k M) \boxtimes (N^* \otimes_k N)$$

$$(\psi \otimes_R \phi) \otimes_k (m \otimes_R n) \mapsto (\phi \otimes_k m) \boxtimes (\psi \otimes_k n).$$

For any morphisms $f : X \longrightarrow Y, g : U \longrightarrow V$ in $\mathfrak{C}$, by means of (2.7) we have the following diagram

(2.8)

$$(\mathcal{F}(Y) \otimes_R \mathcal{G}(V))^* \otimes_k (\mathcal{F}(Y) \otimes_R \mathcal{G}(V))$$

$$\mathrm{id} \otimes_k \mathcal{F}(f) \otimes_k \mathcal{G}(g)$$

$$\cong$$

$$(\mathcal{F}(Y)^* \otimes_k \mathcal{F}(Y)) \boxtimes (\mathcal{G}(Y)^* \otimes_k \mathcal{G}(V))$$

$$(\mathcal{F}(Y) \otimes_R \mathcal{G}(V))^* \otimes_k (\mathcal{F}(X) \otimes_R \mathcal{G}(U)) \qquad L_0 \boxtimes K_0$$

$$(\mathcal{F}(f) \otimes_k \mathcal{G}(g))^* \otimes_k \mathrm{id} \qquad (\mathcal{F}(X)^* \otimes_k \mathcal{F}(X)) \boxtimes (\mathcal{G}(U)^* \otimes_k = G(U))$$

$$\cong$$

$$(\mathcal{F}(X) \otimes_R \mathcal{G}(U))^* \otimes_k (\mathcal{F}(X) \otimes_R = G(U))$$

Using the right exactness of the tensor product we see that $L \boxtimes K$ is the maximum quotient of $L_0 \otimes K_0$ that makes all the above diagrams commutative. The claim of the proposition follows. ∎

Remark: One can easily generalize the above proposition for more functors.

Assume now that $\mathcal{F} : \mathfrak{C} \to R\text{-}\mathsf{Bimod}$ is a monoidal functor, which means there exists an R-bilinear natural isomorphism

$$\theta_{X,Y} : \mathcal{F}(X) \otimes_R \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$$

satisfying the following identity (we assume for simplicity that $\mathfrak{C}$ is strict, i.e., the structure morphisms are identity morphisms)

(2.9)
$$\mathcal{F}(X) \otimes_R \mathcal{F}(Y) \otimes_R \mathcal{F}(Z) \xrightarrow{\theta_{X,Y} \otimes_R \mathrm{id}_{\mathcal{F}(Z)}} \mathcal{F}(X \otimes Y) \otimes_R \mathcal{F}(Z)$$

$$\downarrow{\mathrm{id}_{\mathcal{F}(X)} \otimes_R \theta_{Y,Z}} \qquad\qquad \downarrow{\theta_{X \otimes Y,Z}}$$

$$\mathcal{F}(X) \otimes_R \mathcal{F}(Y \otimes Z) \xrightarrow{\theta_{X,Y \otimes Z}} \mathcal{F}(X \otimes Y \otimes Z)$$

and there exists an isomorphism $\eta : \mathcal{F}(I) \to R$ (I denotes the unit object in $\mathfrak{C}$) satisfying

(2.10)
$$\theta_{I,X} = \eta \otimes_R \mathrm{id}_X, \quad \theta_{X,I} = \mathrm{id}_X \otimes_R \eta$$

It follows easily from definition that monoidal functors preserve rigidity. In fact we can always choose $\mathrm{ev}_{\mathcal{F}(X)}$ and $\mathrm{db}_{\mathcal{F}(X)}$ to be $\mathcal{F}(\mathrm{ev}_X)$ and $\mathcal{F}(\mathrm{db}_X)$, respectively, in case X is (left) rigid.

THEOREM 2.2.4: *Let $\mathfrak{C}$ be a (strict) monoidal category and $\mathcal{F} : \mathfrak{C} \longrightarrow R\text{-}\mathsf{Bimod}$ be a monoidal functor with image in the subcategory of left rigid bimodules. Let $L = \mathsf{Coend}(\mathcal{F})$. Then L is a bialgebroid. If $\mathfrak{C}$ is left rigid, then L is a Hopf algebroid. If $\mathfrak{C}$ is right rigid, then L is an opposite Hopf algebroid.*

Proof. We first show that L is an R-bialgebroid. The product on L is defined as follows. Consider the natural transformation

$$\mathcal{F}(X) \otimes_R \mathcal{F}(Y) \longrightarrow \mathcal{F}(X \otimes Y) \longrightarrow \mathcal{F}(X \otimes Y) \otimes_R L \longrightarrow \mathcal{F}(X) \otimes_R \mathcal{F}(Y) \otimes_R L.$$

According to Proposition 2.2.3, this natural transformation corresponds to a morphism $m : L \boxtimes L \to L$, which according to Lemma 2.1.1 is a morphism of R-coalgebroids. In other words, by means of the diagram in (2.8), m is the unique map $L \boxtimes L \to L$ which satisfies the following diagram for all $X, Y \in \mathfrak{C}$:

$$(2.11) \qquad \begin{array}{ccc} (\mathcal{F}(X) \otimes_R \mathcal{F}(Y))^* \otimes_k (\mathcal{F}(X) \otimes_R \mathcal{F}(Y)) & \longrightarrow & L \\ {\scriptstyle\cong}\downarrow & & \uparrow{\scriptstyle m} \\ (\mathcal{F}(X)^* \otimes_k \mathcal{F}(X)) \boxtimes (\mathcal{F}(Y)^* \otimes_R \mathcal{F}(Y)) & \longrightarrow & L \boxtimes L. \end{array}$$

Further, since $\mathcal{F}(I) \cong R$, R is a comodule over L. The coaction $R \to R \otimes_R L$ yields a morphism of R-coalgebroids $u : R \otimes_k R \to L$. It is easy to deduce from equations (2.9) and (2.10) and the universal property of L the associativity of m and the unital property of u. Thus L is an R-bialgebroid.

Assume that $\mathfrak{C}$ is left rigid. We shall construct the antipode. Recall that L is a quotient of L_0, which is the direct sum of $\mathcal{F}(X)^* \otimes_R \mathcal{F}(X)$. Set $M := \mathcal{F}(X)$. For an element $\varphi \otimes_k m$ of $M^* \otimes M$ we shall use the same notation to denote its image in L. Next, recall that the defining relations for L are obtained from morphism in $\mathfrak{C}$. In particular, we deduce from the canonical morphism $\mathrm{ev}_X : X^* \otimes X \to I$ the following relation on L. Notice that

$$\mathrm{ev}_X{}^* : I \to (X^* \otimes X)^* \cong X^* \otimes X^{**}$$

is nothing but $\mathrm{db}_{X^*} : I \to X^* \otimes X^{**}$. By means of (2.3) for the morphism ev_X

and using (2.7), we have the following commutative diagram, where $M := \mathcal{F}(X)$,

$$(2.12)$$

$$
\begin{array}{ccc}
R \otimes_k (M^* \otimes_k M) & \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_M} & R \otimes_k R \\
{\scriptstyle \mathrm{ev}_M * \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle (s \otimes t)} \\
(M^* \otimes_R M^{**}) \otimes_k (M^* \otimes_R) & \longrightarrow & L \\
{\scriptstyle \cong} \downarrow & & \uparrow {\scriptstyle m} \\
(M^{**} \otimes_k M^*) \boxtimes (M^* \otimes_k M) & \longrightarrow & L \boxtimes L.
\end{array}
$$

Let $\{\varphi_j\}$, $\{\eta^j\}$ be dual bases with respect to db_{M^*}, that is $\mathrm{db}_{M^*} = \sum_j \varphi_j \otimes \eta^j \in M^* \otimes M^{**}$ (see Subsection 1.3). Then (2.12) amounts to the following relation

$$(2.13) \qquad \sum_j (\eta^j \otimes_k \varphi) \circ (\varphi_j \otimes_k m) = \tau(\varphi(m))1,$$

where 1 denotes the unit element in L and $\circ$ denotes the product on L. Similarly, by using the morphism $\mathrm{db}_X : I \to X \otimes X^*$ we obtain the following relation on L

$$(2.14) \qquad (\varphi \otimes_k m_i) \circ (\eta \otimes_k \varphi^i) = \sigma(\eta(\varphi))1,$$

where $\{m_i\}$ and $\{\varphi^i\}$ are dual bases with respect to the map

$$\mathrm{db}_M : R \to M \otimes_R M^* (M = \mathcal{F}(X))), \quad \varphi \in M^*, \quad \eta \in M^{**}.$$

We define now the antipode ∇. Recall that the map $\mathrm{db}_{k,M} : k \to M \otimes_R M^*$ was defined in Subsection 1.3 by $\mathrm{db}_{k,M}(1) = \sum_i m_i \otimes \varphi^i$. Define the map ∇_X for $M = \mathcal{F}(X)$

$$(2.15)$$

$$
\begin{array}{ccc}
M^* \otimes_k M & \xrightarrow{\nabla_X} & L \, {}^\sigma\!\otimes_\sigma L \\
{\scriptstyle \mathrm{id} \otimes \mathrm{db}_{k,M^*} \otimes \mathrm{id}} \downarrow & & \uparrow \\
M^* \otimes_k M^* \otimes_R M^{**} \otimes_k M & \xrightarrow[\cong]{} & (M^{**} \otimes_k M^*) \, {}^\sigma\!\otimes_\sigma (M^* \otimes_k M),
\end{array}
$$

where, in the tensor product $M^{**} \otimes_k M^*$, we use the convention that the action on $M^* = \mathcal{F}(X^*)$ is denote by τ and the action on M^{**} is denoted by σ. It is

straightforward to check the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{F}(Y)^* \otimes_k \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)^* \otimes \mathrm{id}} & \mathcal{F}(X)^* \otimes_k \mathcal{F}(X) \\ \downarrow{\scriptstyle \mathrm{id} \otimes \mathcal{F}(f)} & & \downarrow{\scriptstyle \nabla_X} \\ \mathcal{F}(Y)^* \otimes_k \mathcal{F}(Y) & \xrightarrow[\nabla_Y]{} & L \,{}^{\sigma}\!\otimes_{\sigma} L. \end{array}$$

Thus the universal property of L yields a morphism $\nabla : L \to L \,{}^{\sigma}\!\otimes_{\sigma} L$ which we will show to be the antipode of L. Explicitly we have

$$(2.16) \qquad \nabla(\varphi \otimes m) = (\eta^j \otimes_k \varphi) \,{}^{\sigma}\!\otimes_{\sigma} (\varphi_j \otimes_k m),$$

where the dual bases $\{\eta^j, \varphi_j\}$ are defined above. Now, the equation (1.25), (1.26) for ∇ can be easily deduced from (2.13), (2.14). Let us show (1.25) for $h = \varphi \otimes_k m$. The left hand side of (1.25) is equal to

$$(\eta^j \otimes_k \varphi) \circ (\varphi_j \otimes_k m_i {}_{\tau}\!\otimes^{\sigma} (\varphi^i \otimes_k m)) = \varphi(m_i)(\varphi^i \otimes_k m) = \varphi \otimes_k m,$$

where in the first equality we used (2.14). We thus showed that L is an R-Hopf algebroid.

If $\mathfrak{C}$ is right rigid (in this case the image of $\mathcal{F}$ lies in the subcategory of rigid bimodules), the opposite antipode is induced from the maps

$$\begin{array}{ccc} M^* \otimes_k M & \xrightarrow{\bar{\nabla}_M} & L \,{}_{\tau}\!\otimes^{\tau} L \\ \downarrow{\scriptstyle \mathrm{id} \otimes \mathrm{db}_{*M} \otimes \mathrm{id}} & & \uparrow \\ M^* \otimes_k {}^*M \otimes_R M \otimes_k M & \xrightarrow[\cong]{} & (M \otimes_k {}^*M) \,{}_{\tau}\!\otimes^{\tau} (M^* \otimes_k M), \end{array}$$

where $M = \mathcal{F}(X)$, $X \in \mathfrak{C}$. ∎

COROLLARY 2.2.5: *Let $\mathfrak{C}$ be a small locally finite k-linear abelian monoidal category and $\mathcal{F} : \mathfrak{C} \longrightarrow R$-Bimod be a faithful exact, monoidal functor with image in the subcategory of right f.g. projective modules. Let $L = \mathsf{Coend}(\mathcal{F})$. Then L is semi-transitive coring with respect to the actions (τ, σ) and $\mathcal{F}$ induces a monoidal equivalence between $\mathfrak{C}$ and comod-L. Conversely, let L be a bialgebroid, semi-transitive as a coring with respect to the actions (τ, σ), and $\mathcal{F}$ be the forgetful functor into the category of R-bimodules. Then $\mathcal{F}$ has image in the subcategory of left rigid bimodules and $L \cong \mathsf{Coend}(\mathcal{F})$.*

Proof. We first notice that if $\mathcal{F} : \mathfrak{C} \to \mathfrak{D}$ is at the same time a monoidal functor and an equivalence, then $\mathcal{F}$ is a monoidal equivalence (i.e. the quasi-inverse to

$\mathcal{F}$ is also a monoidal functor). Indeed, let θ and η be the structure morphism for $\mathcal{F}$ as in (2.9) and (2.10). By definition of the quasi-inverse we have the natural isormophims $\mathcal{FG}(U) \cong U$ and $\mathcal{GF}(X) \cong X$, cf. [10, Section IV.4]. Then (still assuming the categories to be strict for simplicity) we define the monoidal functor structure for the quasi-inverse $\mathcal{G}$ of $\mathcal{F}$ as follows:

$$(2.17) \qquad \zeta_{U,V} : \mathcal{G}(U) \otimes \mathcal{G}(V) \cong \mathcal{GF}(\mathcal{G}(U) \otimes \mathcal{G}(V)) \xrightarrow{\mathcal{G}(\theta^{-1})} \mathcal{G}(U \otimes V)$$

$$(2.18) \qquad\qquad\qquad \xi : \mathcal{G}(I_{\mathfrak{D}}) \xrightarrow{\eta^{-1}} \mathcal{GF}(I_{\mathfrak{C}}) \cong I_{\mathfrak{C}}.$$

Assume that we have $\mathcal{F} : \mathfrak{C} \longrightarrow R\text{-Bimod}$ as required. Let L be the Coend of $\mathcal{F}$. Then, by virtue of Theorem 2.1.2, L is semi-transitive as an R-coring with respect to the pair of actions σ, τ and the induced functor $\bar{\mathcal{F}}$ is an equivalence of abelian categories.

On the other hand, by virtue of Theorem 2.2.4, L is an R-bialgebroid and $\bar{\mathcal{F}}$ is a monoidal functor to the category of L-comodules, thus $\bar{\mathcal{F}}$ is a monoidal equivalence.

Assume that the bialgebroid L is semi-transitive coring with respect to the actions (τ, σ). Then the forgetful functor has image in the subcategory of rigid bimodules. Let L' be the Coend of this functor, we have a morphism of bialgebroids $L' \longrightarrow L$ which is an isomorphism of corings, by virtue of Theorem 2.1.2, hence $L' \cong L$ as bialgebroids. ∎

In what follows we will consider only Hopf algebroids with opposite antipode.

Definition (Semi-transitive Hopf algebroids): A Hopf algebroid H is said to be semi-transitive over R if the following conditions are satisfied:

(i) H is semi-transitive as an R-coring with respect to the actions (σ, τ).
(ii) an H-comodule is left rigid as an R-bimodule if and only if it is right rigid.

THEOREM 2.2.6: *Let $\mathfrak{C}$ be a small locally finite k-linear abelian rigid monoidal category and $\mathcal{F} : \mathfrak{C} \longrightarrow R\text{-Bimod}$ be a faithful exact, monoidal functor. Let $H = Coend(\mathcal{F})$. Then H is a semi-transitive Hopf algebroid (with opposite antipode) and $\mathcal{F}$ induces a monoidal equivalence between $\mathfrak{C}$ and finitely generated (over R) right H-comodules. Conversely, let H be a semi-transitive Hopf algebroid and $\mathcal{F}$ be the forgetful functor from the category of finitely generated (over R) H-comodules to $R\text{-Bimod}$. Then $H \cong Coend\mathcal{F}$.*

Proof. $H = \mathsf{Coend}(\mathcal{F})$ is obviously a Hopf algebroid with opposite antipode. The equivalence is established by the corollary above. Also, from the construction, we see that H is a semi-transitive as a coring with respect to the actions (σ, τ).

It remains to show that an H-comodule is left rigid if and only if it is right rigid. Let M be a right H-comodule which left rigid as R-bimodule. Then by the equivalence, $M \cong \mathcal{F}(X)$ for a certain $X \in \mathfrak{C}$. Hence M is rigid since X is rigid. Conversely, if M is a right rigid R-bimodule, then $^*M$ is left rigid. The opposite antipode $^*M$ has a structure of H-comodule, hence $^*M \cong \mathcal{F}(Y)$ for a certain Y in $\mathfrak{C}$. Therefore $M \cong \mathcal{F}(Y^*)$; hence rigid.

Now assume that H is a semi-transitive Hopf algebroid. Then the forgetful functor $\mathcal{F} : \mathsf{comod}\text{-}H \longrightarrow R\text{-}\mathsf{Bimod}$ has image in the subcategory of rigid bimodules. Since H is semi-transitive, this functor is exact (and obviously faithful being forgetful functor). Thus, we can reconstruct the Coend of this functor. From Theorem 2.1.2, $\mathsf{Coend}(F) \cong H$ as corings; hence they are isomorphic as Hopf algebroids. ∎

Remark: The condition (ii) in the definition of semi-transitive Hopf algebroid is not natural. In fact, it is used only for the formulation of Theorem 2.2.6. In other words, Theorem 2.2.6 states that one can "fully" reconstruct a Hopf algebroid from a faithful, exact monoidal functor, in the sense that if we repeat this process we will obtain the same Hopf algebroid. However, we do not have a good criterion for a Hopf algebroid to be reconstructible from its category of comodules. The reader is also referred to [1] for some problems related to the notion of transitivity.

Theorem 2.2.6 has an interesting consequence on characterizing abstract rigid monoidal categories. First, we mention a result of [4].

Let $\mathfrak{C}$ be a small abelian rigid monoidal category with an injective cogenerator. Then there exists an exact faithful monoidal functor $\mathfrak{C} \longrightarrow R\text{-}\mathsf{Bimod}$ for a certain ring R.

By using the above result of reconstruction and representation, we can easily deduce the following result

COROLLARY 2.2.7: *Let $\mathfrak{C}$ be a small k-linear locally finite abelian rigid monoidal category with an injective cogenerator. Then there exists a ring R such that $\mathfrak{C}$*

is monoidally equivalent with the category of f.g. projective R-comodules over a certain semi-transitive Hopf algebroid over R.

References

[1] A. Bruguières, *Théorie tannakienne non commutative*, Communications in Algebra **22** (1994), 5817–5860.

[2] P. Deligne, *Catégories tannakiennes*, in *The Grothendieck Festschrift (P. Cartier and et. al., eds.)*, Progr. Math., 87, vol. II, Birkhäuser Boston, Boston, MA, 1990, pp. 111–195.

[3] S. Doplicher and J. E. Roberts, *A new duality theory for compact quantum groups*, Inventiones Mathematicae **98** (1989), 157–218.

[4] Phung Ho Hai, *An embedding theorem of abelian monoidal categories*, Compositio Mathematica **132** (2002), 27–48; Corrigendum to appear.

[5] T. Hayashi, *Compact quantum groups of face type*, Kyoto University, Research Institute for Mathematical Sciences Publications **32** (1996) 351-369.

[6] T. Hayashi, *Quantum Groups and Quantum Semigroups*, Journal of Algebra **204** (1998).

[7] J.-H. Lu, *Hopf algebroids and quantum groupoids*, International Journal of Mathematics **7** (1996), 47–70.

[8] V. V. Lyubashenko, *Hopf Algebras and Vector Symmetries*, Russian Mathematical Survey **41** (1986), no. 5, 153–154.

[9] V. V. Lyubashenko, *Square Hopf algebras*, Memoir of AMS **142** (1999).

[10] S. MacLane, *Categories, for the Working Mathematician*, Springer Verlag, New York-Berlin, 1971.

[11] S. Majid, *Algebras and Hopf Algebras in Braided Categories*, in *Advances in Hopf Algebras, LN Pure and Applied Mathematics*, vol. 158, 1994, pp. 55–105.

[12] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995.

[13] G. Maltsiniotis, *Groupoïde Quantiques*, Paris Comptes Rendus Mathématique, Académie des Sciences **314** (1992), 249–252.

[14] P. McCrudden, *Categories of representations of coalgebroids*, Advances in Mathematics **154** (2000), 299–332.

[15] B. Pareigis, *Reconstructions of Hidden-Symmetries*, Journal of Algebra **183** (1996), 90–154.

[16] R. N. Saavedra, *Catégories Tannakiaennes*, vol. 265, *Lecture notes in mathematics*, Springer Verlag, Berlin–New York, 1972.

[17] P. Schauenburg, *Duals and Doubles of Quantum Groupoids ($\times_R$-Hopf Algebras)*, Contemporary Mathematics **267** (2000), 273–299.

[18] P. Schauenburg, *The monoidal center construction and bimodules*, Journal of Pure and Applied Algebra **158** (2001), 325–346.

[19] M. Takeuchi, *Groups of algebras over $A \otimes \bar{A}$*, Journal of the Mathematical Society of Japan **29** (1977), 459–492.

[20] M. Takeuchi, *$\sqrt{Morita}$ theory*, Journal of the Mathematical Society of Japan **39** (1987), 301–336.

[21] S. L. Woronowicz, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups*, Inventiones Mathematicae **93** (1988), 35–76.

[22] S. L. Woronowicz, *Compact matrix pseudogroups*, Communications in Mathematical Physics **111** (1987), 613–665.

[23] Ping Xu, *Quantum groupoids*, Communications in Mathematical Physics **216** (2001), 539–581.